# Low energy limit of QCD and the emerging of confinement

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# Plan of the talk

- Classical field theory
  - Scalar field theory
  - Yang-Mills theory
  - Yang-Mills Green function
- Quantum field theory
  - Scalar field theory
  - Yang-Mills theory
  - QCD in the infrared limit
  - Bosonization
  - Instantons
  - $\sigma$ mass
  - Confinement
- Conclusions

# Classical field theory: Scalar field

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being sn an elliptic Jacobi function and $\mu$ and $\theta$ two constant. This solution holds provided the following dispersion relation holds

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- Mass arises from the nonlinearities when $\lambda$ is taken to be finite rather than going to zero.

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- One can prove that this is indeed so provided

$$\delta\phi = \kappa^2\lambda \int d^4x' d^4x'' G(x-x')[G(x'-x'')]^3 j(x') + O(j(x)^3)$$

  with the identification $\kappa = \mu$, the same of the exact solution, and $\Box G(x-x') + \lambda[G(x-x')]^3 = \mu^{-1}\delta^4(x-x')$.

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- This implies that the corresponding quantum field theory, in a very strong coupling limit, takes a Gaussian form and is trivial (triviality of the scalar field theory in the infrared limit).
- All we need now is to find the exact form of the propagator $G(x-x')$ and we have completely solved the classical theory for the scalar field in a strong coupling limit.

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- In order to solve the equation

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- It is straightforwardly obtained the Fourier transformed solution

$$G_0(\omega)=\sum_{n=0}^{\infty}(2n+1)\frac{\pi^2}{K^2(i)}\frac{(-1)^n e^{-(n+\frac{1}{2})\pi}}{1+e^{-(2n+1)\pi}}\frac{1}{\omega^2-m_n^2+i\epsilon}$$

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- We are able to recover the fully covariant propagator by boosting from the rest reference frame obtaining finally

$$G(p) = \sum_{n=0}^{\infty}(2n+1)\frac{\pi^2}{K^2(i)}\frac{(-1)^n e^{-(n+\frac{1}{2})\pi}}{1+e^{-(2n+1)\pi}}\frac{1}{p^2 - m_n^2 + i\epsilon}.$$

This shows that our solution given above indeed represents a strong coupling expansion being meaningful for $\lambda \to \infty$.

# Classical field theory: Yang-Mills field (1)

- A classical field theory for the Yang-Mills field is given by

$$\partial^\mu\partial_\mu A^a_\nu-\left(1-\frac{1}{\xi}\right)\partial_\nu(\partial^\mu A^a_\mu)+gf^{abc}A^{b\mu}(\partial_\mu A^c_\nu-\partial_\nu A^c_\mu)+gf^{abc}\partial^\mu(A^b_\mu A^c_\nu)+g^2f^{abc}f^{cde}A^{b\mu}A^d_\mu A^e_\nu=-j^a_\nu.$$

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- For the homogeneous equation, we want to study it in the formal limit $g\rightarrow\infty$. We note that a class of exact solutions exists if we take the potential $A^a_\mu$ just depending on time, after a proper selection of the components [see Smilga (2001)]. These solutions are the same of the scalar field when spatial coordinates are set to zero (rest frame).

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- Differently from the scalar field, we cannot just boost away these solutions to get a general solution to Yang-Mills equations due to gauge symmetry. But we can try to find a set of similar solutions with the proviso of a gauge choice.
- This kind of solutions will permit us to prove that a set of them exists supporting a trivial infrared fixed point to build on a quantum field theory.

# Classical field theory: Yang-Mills field (2)

- Exactly as in the case of the scalar field we assume the following solution to our field equations

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- This implies that the corresponding quantum theory, in a very strong coupling limit, takes a Gaussian form and is trivial.
- The crucial point, as already pointed out in the eighties [T. Goldman and R. W. Haymaker (1981), T. Cahill and C. D. Roberts (1985)], is the exact determination of the gluon propagator in the low-energy limit. This will determine completely low-energy physics for strong interactions

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- By direct substitution into Yang-Mills equations one recovers the equation for $\phi$ that is

$$\partial^\mu \partial_\mu \phi - \frac{1}{N^2-1}\left(1-\frac{1}{\xi}\right)(\eta^a \cdot \partial)^2 \phi + N g^2 \phi^3 = -j_\phi$$

being $j_\phi = \eta_\mu^a j^{\mu a}$ and use has been made of the formula $\eta^{\nu a} \eta_\nu^a = N^2 - 1$.

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- In the Landau gauge (Lorenz gauge classically) this equation is exactly that of the scalar field given before and we get again a current expansion also for the scalar field.
- So, a set of solutions of the Yang-Mills equations exists supporting a trivial infrared fixed point. Our aim is to study the theory in this case.

# Yang-Mills-Green function

- The instanton solutions given above permit us to write down immediately the propagator for the Yang-Mills equations in the Landau gauge for SU(N) being exactly the same given for the scalar field:

$$\Delta^{ab}_{\mu\nu}(p)=\delta_{ab}\left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\sum_{n=0}^{\infty}\frac{B_n}{p^2-m_n^2+i\epsilon}+O\left(\frac{1}{\sqrt{N}g}\right)$$

being

$$B_n=(2n+1)\frac{\pi^2}{K^2(i)}\frac{(-1)^{n+1}e^{-(n+\frac{1}{2})\pi}}{1+e^{-(2n+1)\pi}}$$

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- The constant $\Lambda$ must be the same constant that appears in the ultraviolet limit by dimensional transmutation, here arises as an integration constant [M. Frasca, arXiv:1007.4479v2 [hep-ph]].
- This is the propagator of a massive field theory but the mass poles arise dynamically from the non-linearities in the equations of motion. At this stage we are working classically yet.

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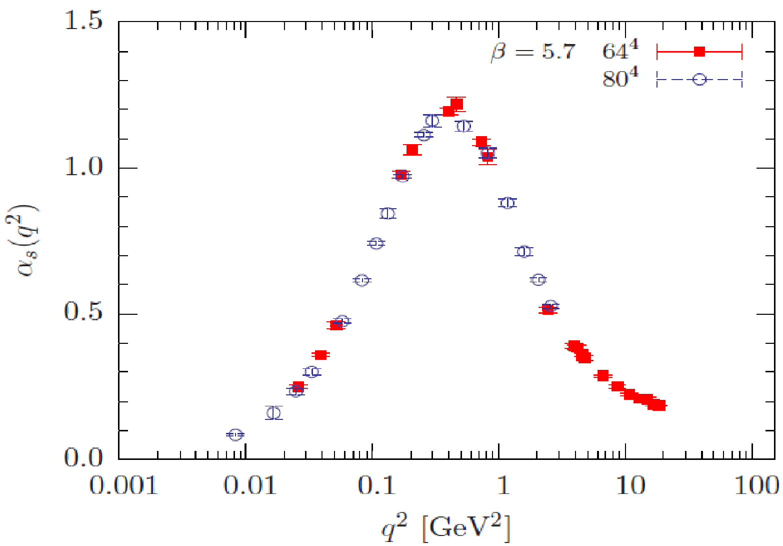

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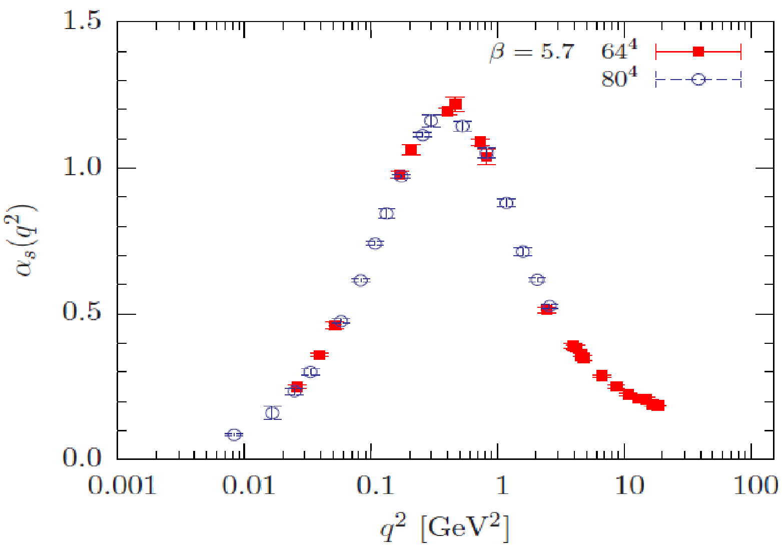


- A similar result was also obtained by Boucaud et al. [“The strong coupling constant at small momentum as an instanton detector “, JHEP 0304, 005 (2003)] again with lattice computations.

# Quantum field theory: Scalar field (1)

- We can formulate a quantum field theory for the scalar field starting from the generating functional

$$Z[j]=\int[d\phi]\exp\left[i\int d^4x\left(\frac{1}{2}(\partial\phi)^2-\frac{\lambda}{4}\phi^4+j\phi\right)\right].$$

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  Then we can seek for a solution series as $\phi=\sum_{n=0}^{\infty}\lambda^{-n}\phi_n$ and rescale the current $j\rightarrow j/\lambda$ being this arbitrary.

- It is not difficult to see that the leading order correction can be computed solving the classical equation

$$\Box\phi_0+\phi_0^3=j$$

  that we already know how to manage. This is completely consistent with our preceding formulation [M. Frasca (2006)] but now all is fully covariant. We are just using our ability to solve the classical theory.

# Quantum field theory: Scalar field (2)

- Using the approximation holding at strong coupling

$$\phi_0 = \mu \int d^4x G(x-x')j(x') + \ldots$$

it is not difficult to write the generating functional at the leading order in a Gaussian form

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- This conclusion is really important: It says that the scalar field theory in d=3+1 is <u>trivial</u> in the infrared limit!
- This functional describes a set of free particles with a mass spectrum

$$m_n = (2n+1)\frac{\pi}{2K(i)}\left(\frac{\lambda}{2}\right)^{\frac{1}{4}}\mu$$

that are the poles of the propagator, the ones of the classical theory.

# Quantum field theory: Yang-Mills field (1)

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$$D^{ab}_{\mu\nu}(p)=\delta_{ab}\left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\sum_{n=0}^\infty\frac{B_n}{p^2-m_n^2+i\epsilon}+O\left(\frac{1}{\sqrt{N}g}\right)$$

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- The next step is to use the approximation that holds in a strong coupling limit

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- and we note that, in this approximation, the ghost field just decouples and becomes free and one finally has at the leading order

$$Z_0[j]=N\exp\left[\frac{i}{2}\int d^4x'd^4x''j^{a\mu}(x')D^{ab}_{\mu\nu}(x'-x'')j^{b\nu}(x'')\right].$$

This functional describes free massive glueballs that are the proper states in the infrared limit. Yang-Mills theory is <u>trivial</u> in the limit of the coupling going to infinity and we expect the running coupling to go to zero lowering energies.

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- Indeed, we will have

$$S_g = -\int d^4x \left[\bar{c}^a \partial_\mu \partial^\mu c^a + O\left(\frac{1}{\sqrt{N}g}\right) + O\left(j^3\right)\right]$$

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- Now, we can take a look at the ghost part of the action. We just note that, for this particular form of the propagator, inserting our approximation into the action produces an action for a free ghost field.
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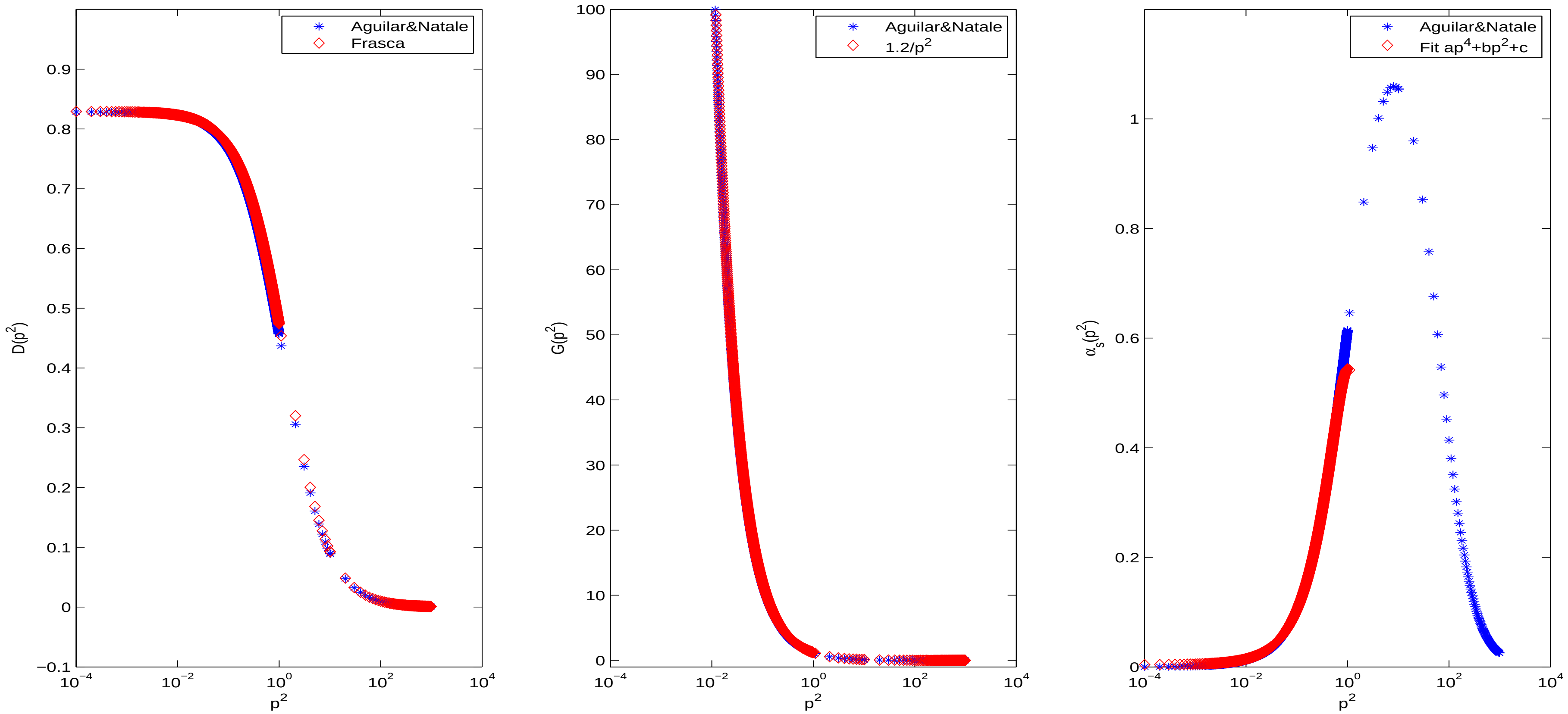
- Our conclusion is that, in a strong coupling expansion $1/\sqrt{N}g$, we get the so called <u>decoupling solution</u>.

# Quantum field theory: Yang-Mills field (3)

A direct comparison of our results with numerical Dyson-Schwinger equations gives the following:




that is strikingly good (ref. A. Aguilar, A. Natale, JHEP 0408, 057 (2004)).

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- We recognize here an explicit Yukawa interaction and a Nambu-Jona-Lasinio non-local term. Already at this stage we are able to recognize that NJL is the proper low-energy limit for QCD at zero temperature.

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- Now we operate the Smilga's choice $\eta^a_\mu\eta^b_\nu = \delta_{ab}(\eta_{\mu\nu} - p_\mu p_\nu/p^2)$ for the Landau gauge.

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- We want to give explicitly the contributions from gluon resonances. In order to do this, we introduce the bosonic currents $j_\mu^a(x) = \eta_\mu^a j(x)$ with the current $j(x)$ that of the gluonic excitations after mapping.

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- Now, we recognize that the propagator is just a sum of Yukawa propagators weighted by exponential damping terms. So, we introduce the $\sigma$ field and truncate at the first excitation. This is a somewhat rough approximation but will be helpful in the following analysis.
- This means the we can write the bosonic currents contribution as coming from a boson field and written down as $\sigma(x)=\sqrt{3(N_c^2-1)/B_0}\int d^4x'\Delta(x-x')j(x')$.

# QCD at the infrared limit (4)

- So, the model we consider for our finite temperature analysis, directly derived from QCD, is [Weise et al., Phys. Rev. D79, 014022 (2009), arXiv:0810.1099v2 [hep-ph]]

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- Now, we recover the non-local model of Weise et al. directly from QCD ($2\mathcal{G}(0) = G$ is the standard NJL coupling)

$$\mathcal{G}(p) = -\frac{1}{2}g^2\sum_{n=0}^{\infty}\frac{B_n}{p^2-(2n+1)^2(\pi/2K(i))^2\sigma+i\epsilon} = \frac{G}{2}\mathcal{C}(p)$$

with $\mathcal{C}(0) = 1$ fixing in this way the value of $G$ using the gluon propagator.

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- This holds together with the gap equations

$$M(p) = m_q + \mathcal{C}(p)v$$

$$v = 4G_{eff}NN_f \int \frac{d^4p}{(2\pi)^4}\mathcal{C}(p)\frac{M(p)}{p^2 + M^2(p)}$$

where we can identify a $G_{eff} = \frac{1}{m_0^2 + 1/G} < G$ due to the mass gap $m_0$.

# Instanton liquid

- For aims of completeness, we give here a comparison of our gluon propagator (the form factor) with the one used in Weise et al. based on an instanton liquid model and the one derived for an instanton liquid [T. Schäfer and E. V. Shuryak, Rev. Mod. Phys. 70, 323 (1998)].

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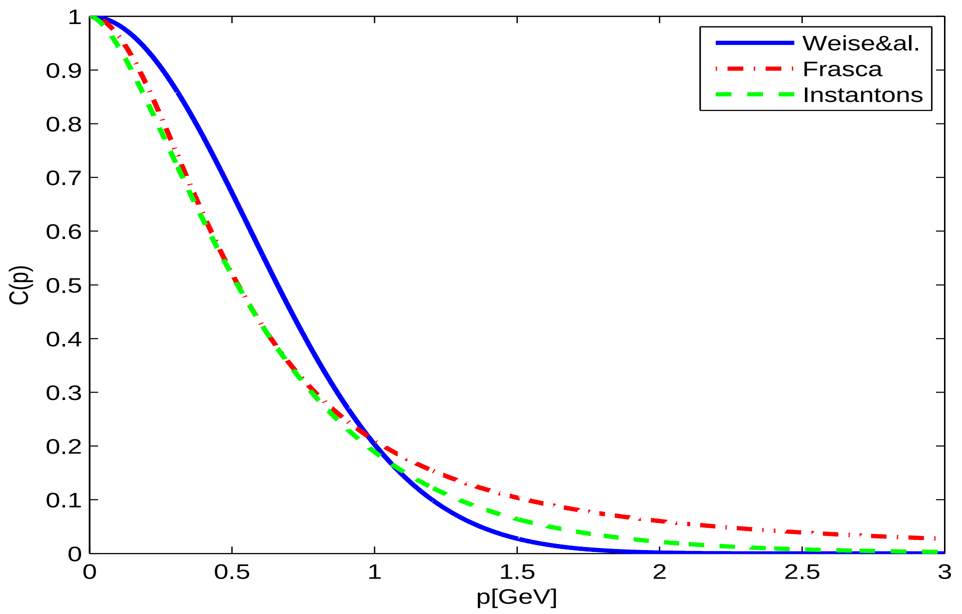

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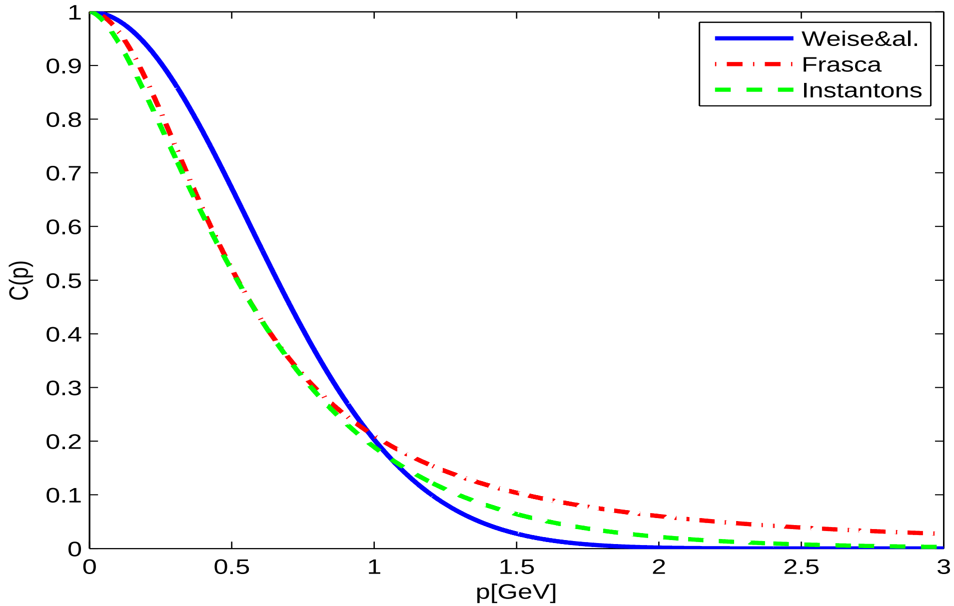


- Istanton liquid approximation is a good one indeed in describing the ground state of Yang-Mills theory!

# $\sigma$ mass

- In NJL $\sigma$ mass is given by $m_\sigma = \sqrt{4m^{*2} + m_\pi^2}$ being $m^*$ the quark constituent mass obtained from the gap equation of the model. In our case we have $m^* = 214\ MeV$ and $m_\pi = 139.7\ MeV$ with a 4d cut-off.

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- Analysis on experimental data gives for this particle [Caprini, Colangelo, Leutwyler, PRL 96, 132001 (2006)]

$$m_\sigma = 441^{+16}_{-8}\ MeV$$

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- Our model provides

$$m_\sigma = 451 \pm 20\ MeV$$

the error arising from string tension, in close agreement with these results. This permits us to conclude that $\sigma$ particle is a glue particle arising from the Yang-Mills part of the QCD Lagrangian, in agreement with recent studies [e.g. G. Mennessier, S. Narison, X.-G. Wang, PLB696, 40 (2011) and refs. therein.].

# Confinement (1)

- In order to evaluate confinement in a pure Yang-Mills theory we have to evaluate [P. González, V. Mathieu, and V. Vento, PRD 84, 114008 (2011)]:

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$$\alpha_s(p) = \alpha_s(0)\frac{p^4}{\Lambda^4}.$$

- This implies for the potential

$$V(r) = -\frac{\alpha_s(0)}{\Lambda^4}\frac{\partial^4}{\partial r^4}\sum_{n=0}^{\infty}(2n+1)\frac{\pi^2}{K^2(i)}\frac{(-1)^n e^{-(n+\frac{1}{2})\pi}}{1+e^{-(2n+1)\pi}}\frac{e^{-m_n r}}{r}$$

and due to massive excitations one gets a screened potential. This appears to agree very well with the conclusions given in [P. González, V. Mathieu, and V. Vento, PRD 84, 114008 (2011)] but not in agreement with Cornell potential observed on the lattice for quenched simulations.

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- This result is in agreement with expectations from Cornwall's analysis [J. M. Cornwall, Phys. Rev. D 26, 1453 (1982)] that the gluon mass gets a dependence on momenta [see also Binosi, Aguilar, Papavassiliou].

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- A set of classical solutions is proved to exist for the Yang-Mills field, instantons, that support the view of a trivial infrared fixed point.
- A low-energy limit of QCD is so obtained that reduces to a non-local Nambu-Jona-Lasinio model with all the parameters and the form factor properly fixed.

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- A set of classical solutions is proved to exist for the Yang-Mills field, instantons, that support the view of a trivial infrared fixed point.
- A low-energy limit of QCD is so obtained that reduces to a non-local Nambu-Jona-Lasinio model with all the parameters and the form factor properly fixed.
- $\sigma$ meson gets a mass in close agreement with recent studies on $\pi\pi$-scattering and its very nature is gluonic and not a tetraquark state.

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