Solving Schrödinger equation via Tartaglia/Pascal triangle: A possible link between stochastic processing and quantum mechanics

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ABSTRACT
In a recent paper [1] it was shown a clean connection between the solution of the classical heat equation and the Tartaglia/Pascal/Yang-Hui triangle. When the time variable in the heat equation is substituted with the imaginary time, the heat equation becomes the Schrödinger equation of the quantum mechanics. So, a conjecture was put forward about a connection between the solution of the Schrödinger equation and a suitable generalization of the Tartaglia triangle. This paper proves that this conjecture is true and shows a new – as far as the authors are aware – result concerning the generalization of the classical Tartaglia triangle by introducing the "complex valued Tartaglia triangle". A "complex valued Tartaglia triangle" is just the square root of an ordinary Tartaglia triangle, with a suitable phase factor calculated via a discretized version of the ordinary continuous case of the Schrödinger equation. So, taking the square of this complex valued Tartaglia triangle gives back exactly the probability distribution of a discrete random walk. We also discuss about potential connections between the theories of stochastic processes and quantum mechanics: a connection debated since the incept of the theories and still lively hot today.

KEYWORDS: Stochastic processing, quantum mechanics, Schrödinger equation, Tartaglia-Pascal triangle.

1. INTRODUCTION

In a recent paper [1] we demonstrated how the Tartaglia-Pascal triangle was ubiquitous in the areas of stochastic processes[1], filtering and finance and indeed how it represents an underlying layer to effects where a noisy behavior is present. From a historical point of view, the path that took toward this key conclusion can be traced back from Fourier [3] to Einstein [4] in 1905 marking the beginning of the modern theory of stochastic processes.

A typical system that appears to behave in a noisy way is a quantum system and so, in [1] authors put forward a conjecture that some kind of Tartaglia-Pascal triangle should be also there even if they were not able to give a clear understanding of this idea. The problem can be traced back to the presence of an imaginary time [5] that makes impossible a connection with a classical stochastic process. Indeed, the spreading of the wave packet of a free particle in time goes proportionally to the square of time rather than linearly as it happens in the diffusion of classical Brownian motion. Notwithstanding such a difficulty, Edward Nelson put forward a relevant connection between stochastic processes and quantum mechanics [6] that was further expanded by Francesco Guerra [7]. But this underwent severe criticisms that were summed up in a paper by Hänggi and coworkers with the fundamental claim that no classical stochastic process can describe quantum mechanics [8].

So, apparently, no direct connection seems to exist between a Tartaglia-Pascal triangle and quantum mechanics. In this paper we will prove that this connection does exist but in a rather subtle way: the triangle appears in quantum mechanics as a suitable square root of the triangle itself. This will represent our main

[1] A beautiful description of stochastic process and stochastic filtering is in [2].

result. The interesting aspect behind this result is a possible formulation of a discrete quantum mechanics [9][10] that will take its start from the discretization of the heat equation, making deeper the connection between a classical stochastic process and quantum mechanics.

This conclusion prompted one of us to show that quantum mechanics can be reformulated as the square root of a classical stochastic process [11]. This entails an extension to the concept of Itō integral.

The remaining part of the paper is organized as follows. Section 2 – the core of the paper – illustrates the new results related to the conjecture made in [1]. It introduces the "complex valued Tartaglia Triangle" by solving the Schrödinger equation for the case of a single particle that is localized with a given uncertainty. Section 2 is a clue that a deep connection between quantum mechanics and stochastic process indeed exists. Section 3 provides a numerical analysis of the theoretical results obtained in the preceding section. In this case, a relevant connection with hyperbolic geometry emerges. Finally, Section 4 provides the conclusions.

2. MAIN RESULTS

Stochastic processes are ubiquitous in nature and, in order to describe them, a mathematical framework was devised that is common knowledge in science and technology. The paradigm of a stochastic process is the Brownian motion. In its simplest form, Brownian motion is equivalent to solve the stochastic differential equation [2]

$$dX(t)=\sigma^2\, dW(t) \tag{1}$$

being $W(t)$ a stochastic process with a normal distribution with zero mean and variance 1. This process entails a diffusion process for the probability density function – pdf (see also Section 8.1 of [1]) that has as evolution equation the heat equation, a particular case of the more general Fokker-Planck equation ([2]) typical of these processes,

$$\frac{\partial u(x,t)}{\partial t}-\gamma\,\frac{\partial^2 u(x,t)}{\partial t^2}=0,\quad \text{for} -\infty < x < \infty \text{ and } 0 < t < \infty \tag{2}$$

being $\gamma=\sigma^2/2$ the diffusion coefficient, u_t the partial derivative of u with respect to t, and u_{xx} the second order partial derivative of u with respect to x. For the sake of simplicity, we consider just the one dimensional case. Higher dimensions can be treated similarly. This equation was well-known since 1800, being initially postulated by Fourier [3], but it was with the fundamental understanding due to Einstein of the Brownian motion, at beginning of 900s, that its role into the realm of stochastic processes was really understood [4].

Some years after the formulation of the theory of Brownian motion, a new paradigm in science took over: quantum mechanics. This theory gives a deep understanding of the dynamics of elementary particles in terms of complex-valued probability amplitudes and so, this theory is deeply rooted on probabilistic modeling. The intriguing aspect of this theory is that at its foundation lays the Schrödinger partial differential equation. It has the property that can be obtained by the heat equation by simply multiplying the time variable by imaginary unity $i=\sqrt{-1}$ (giving rise to the so called imaginary time). So, one can ask if some kind of stochastic process is representing this equation much in the same way this was done for the heat equation. Our main result in this section will give an answer to this question that has had as a pioneer Edward Nelson [6]. Nelson approach was shown to be incorrect in [8]. Indeed, as we will show, quantum mechanics cannot be put in a direct correspondence with Brownian motion, as Nelson expected. Rather this relation is somewhat more intriguing and Tartaglia triangle [1] will be our main tool to understand this.

In this section we fix some notations, give the mathematical basis and put out the main results. Our starting reference is [1] where the asymptotic approximation (as n grows large, for k in the neighbourhood of $0.5n$, [12]) for binomial coefficients is given[2]

$$p_n(k)=\binom{n}{k}\frac{1}{2^n}\approx\frac{1}{\sqrt{2\pi\frac{n}{4}}}e^{-\frac{\left(k-\frac{n}{2}\right)^2}{2\frac{n}{4}}}\equiv\phi\left(\frac{n}{2},\frac{n}{4};k\right) \tag{3}$$

and mapped onto the solution of the heat equation (heat kernel) in the following way

$$\binom{n}{k}\approx 2^n\cdot\phi\left(\frac{n}{2},\frac{n}{4};k\right)=2^n\cdot H\left(x=k-\frac{n}{2},t=\frac{n}{8},\gamma=1\right) \tag{4}$$

provided that

$$H(x,t,\gamma)=\frac{1}{\sqrt{4\pi\gamma t}}e^{-\frac{x^2}{4\gamma t}} \tag{5}$$

is the kernel of the heat equation. Thus a generic solution of heat equation (for the 1D case) is

$$u(x,t)=\int_{-\infty}^{+\infty}H(x-y,t,\gamma)f(y)dy \tag{6}$$

given the initial condition: $u(x,t=0)=f(x)$. Equation (5) is a pdf of Gaussian stochastic process with variance proportional to "*t*". The following Chapman-Kolmogorov equation[3] holds true for this case, provided we identify as a random variable the position in space and use a continuous variable,

$$H(x-y,t_x-t_y)=\int_{-\infty}^{+\infty}H(x-z,t_x-t_z)H(z-y,t_z-t_y)dz \tag{7}$$

and *H(.,.)* represents the pdf of a stochastic process ***a la*** Markov with a variance going like "*t*" at increasing time.

We emphasize the perfect match between the solution of the kernel of the heat equation via the binomial coefficients and, consequently, the Tartaglia triangle elements as explained in Section 7.1 and appendix C of [1].

Heat equation, when the time is Wick-rotated[4] ([5]) on the imaginary axis (i.e. $t\rightarrow it$, imaginary time), becomes the following Schrödinger equation (with all the constants properly inserted for this case)

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[2] The binomial pdf $p_n(k)$ is approximated by a Gaussian pdf with mean value $n/2$ and variance $n/4$.

[3] The Chapman–Kolmogorov equation is an identity relating the joint probability distributions of different sets of coordinates on a stochastic process. (Source: Wikipedia)

[4] Gian Carlo Wick (Turin, 1909-1992) was an Italian physicist. He worked with Heisenberg and Fermi. He is mostly known for important contributions to quantum field theory.

$$-\frac{\eta^2}{2m}\frac{\partial^2\psi}{\partial x^2}=i\eta\frac{\partial\psi}{\partial t} \tag{8}$$

where t is the time variable, x the spatial variable, ħ is the reduced Planck constant, m is the mass of the particle, and Ψ the wave function. It is worth to notice that by defining

$$\gamma=\frac{i\eta}{2m} \tag{9}$$

Eq. (8) is formally identical to Eq. (2) (Heat equation) and, even if shown for 1D case this is truth for the n-dimensional case.

One can ask whether Ψ represents a stochastic process in some way and, eventually, how the asymptotic of the binomial coefficients, i.e. Tartaglia triangle, map onto it. Indeed, when one assumes a particle perfectly localized so that $\psi(x,0)=K(x,0)=\delta(x)$, the following so called "propagator" is obtained

$$K(x,t)=\left(\frac{m}{2\pi\eta it}\right)^{\frac{1}{2}}e^{\frac{i}{\eta}\frac{x^2}{2t}} \tag{10}$$

that satisfies the following equation, analogous to the above Chapman-Kolmogorov equation

$$K(x-z,t_x-t_z)=\int K(x-y,t_x-t_y)K(y-z,t_y-t_z)dy \tag{11}$$

A generic solution of the Schrödinger takes the form

$$\psi(x,t)=\int K(x-y,t)\xi(y)dy \tag{12}$$

where ξ(y) is a given initial wave function.

All this resembles very near the formalism we gave above for the heat equation but now, using Eq. (11), it is not difficult to show that - for a generic solution – time evolution in quantum mechanics does not represent a stochastic process ([8]). Specifically by applying Eq. (12) to wave function it can be shown that (in its more general form) it does not represent a stochastic process.

$$\begin{aligned}&\int\psi(x-y,t_x-t_y)\psi(y-z,t_x-t_y)dy=\\&=\int dy\int K(x-y-x',t_x-t_y)\xi(x')dx'\int K(y-z-x'',t_y-t_z)\xi(x'')dx''=\\&=\int K(x-x'-z+x'',t_x-t_z)\xi(x')\xi(x'')dx'dx''\neq\psi(x-z,t_x-t_z)\end{aligned} \tag{13}$$

This has the notable exception when the particle, at time $t=0$, rather than being perfectly localized (i.e. with a Dirac impulse) is modelled with a Gaussian wave function having variance a^2 and position x_0

$$\xi(x)=\frac{1}{\left(\pi a^2\right)^{\frac{1}{4}}}e^{-\frac{(x-x_0)^2}{2a^2}} \tag{14}$$

The reason to choose a Gaussian wave function, with a normalization such that its square integrated on all volume is 1, is that we aim to compare time evolution in quantum mechanics with that of a Brownian motion that has a typical Gaussian pdf. Once eq. (14) it is time-evolved through eq. (12), we get the wave function (see Appendix A and [15])

$$\psi(x,t)=\frac{1}{\left(2\pi a^2\right)^{\frac{1}{4}}}\frac{1}{\sqrt{1+i\frac{t}{\tau_H}}}e^{-\frac{(x-x_0)^2}{4a^2}\frac{1}{1+i\frac{t}{\tau_H}}} \tag{15}$$

where we have introduced the Heisenberg time $\tau_H = ma^2/\eta$ that is the time scale over which the wave packet spreads during the motion. Then, the pdf of the particle is given by the squared wave function

$$S(x,t,\tau_H,a)=\left|\psi(x,t)\right|^2=\frac{1}{\left(2\pi a^2\right)^{\frac{1}{2}}}\frac{1}{\sqrt{1+\frac{t^2}{\tau_H^2}}}e^{-\frac{(x-x_0)^2}{2a^2}\frac{1}{1+\frac{t^2}{\tau_H^2}}} \tag{16}$$

From this result it easily realized that this quantum process is not directly Markovian as the variance goes like "t^2" at increasing time rather than like "t" confirming again what we showed above. Specifically, notwithstanding the resemblance between the formalism for the heat equation and the Schrödinger equation, (the latter having an imaginary time) it changes significantly the meaning of the results.

In order to catch a connection with binomial coefficients, and so the Tartaglia triangle, we have to move from a continuum space-time formulation of quantum mechanics to a discrete one. Once we have got the link we can turn back to the continuum case. So, to understand what kind of stochastic process could represent the operator of time time evolution in quantum mechanics, we perform an identical mapping of (4) as for the heat equation onto the probability density (16). The following holds

$$\binom{n}{k}\approx 2^n\phi\left(\frac{n}{2},\frac{n}{4};k\right)=2^n S\left(x=x_0+k-\frac{n}{2},t=\sqrt{\frac{n}{4}-1},\tau_H=1,a=1\right) \tag{17}$$

We have just remapped a solution of the Schrödinger equation onto the asymptotic of the binomial coefficients, and so onto the Tartaglia triangle, using the following mapping

$$\begin{aligned} n &= 4(t^2+1) \\ k &= x+2(t^2+1) \end{aligned} \tag{18}$$

Moreover this pushes us to do a further step by noting that, in quantum mechanics, the fundamental mathematical tool is the wave function. So, we repeat the above mapping directly onto the wave function (15) obtaining what we call the "complex-valued Tartaglia amplitudes"

$$T(n,k) = 2^{\frac{n}{2}} \psi\left(x = x_0 + k - \frac{n}{2}, t = \sqrt{\frac{n}{4} - 1} \right)\Bigg|_{\tau_H = 1, a = 1} \tag{19}$$

that are defined in such a way to have

$$|T(n,k)|^2 = 2^n \phi\left(\frac{n}{2}, \frac{n}{4}; k \right) \approx \binom{n}{k} \tag{20}$$

which is again the classical Tartaglia triangle. Complex-valued Tartaglia amplitudes can be written down as

$$T(n,k) \approx \binom{n}{k}^{\frac{1}{2}} e^{i\tilde{\phi}(n,k)} \tag{21}$$

where the phase term $\tilde{\phi}(n,k)$ is derived directly from the Schrödinger equation by considering the mapping of eqs. (18) for eq. (15). *T(n,k)* can be decomposed into an imaginary and a real parts as follows

$$\begin{aligned} x(n,k) &= \binom{n}{k}^{\frac{1}{2}} \cos\left(\tilde{\phi}(n,k)\right) \\ y(n,k) &= \binom{n}{k}^{\frac{1}{2}} \sin\left(\tilde{\phi}(n,k)\right) \end{aligned} \tag{22}$$

Now, from eq. (15) we derive the following phase in the (x,t) continuous domain

$$\theta(x,t) = -\frac{1}{2}\tan^{-1}\left(\frac{t}{\tau_H} \right) + \frac{(x - x_0)^2}{4a^2} \frac{t}{\tau_H} \frac{1}{1 + \frac{t^2}{\tau_H^2}} \tag{23}$$

Thus we obtain the phase for complex-valued Tartaglia amplitudes in (n,k) discrete domain

$$\tilde{\phi}(n,k) = \theta\left(x = x_0 + k - \frac{n}{2}, t = \sqrt{\frac{n}{4} - 1} \right)\Bigg|_{\tau_H = 1, a = 1} = -\frac{1}{2}\tan^{-1}\left(\sqrt{\frac{n}{4} - 1} \right) + \frac{\left(k - \frac{n}{2} \right)^2}{n} \sqrt{\frac{n}{4} - 1} \tag{24}$$

From equations (19), (21) and (24) the Schrödinger equation solution on the (n,k) lattice is

$$\psi(n,k) = 2^{-\frac{n}{2}} \binom{n}{k}^{\frac{1}{2}} e^{i\left[\frac{\left(k - \frac{n}{2} \right)^2}{n} \sqrt{\frac{n}{4} - 1} - \frac{1}{2}\tan^{-1}\left(\sqrt{\frac{n}{4} - 1} \right) \right]} \tag{25}$$

The equations (19), (21), (24) and (25) are the main results of this paper. Now, if we assume a Hilbert space equipped with the scalar product

$$\left\langle \phi_1, \phi_2 \right\rangle = \sum_{k=0}^{n} \phi_1^*(n,k)\phi_2(n,k) \tag{26}$$

we are able to properly normalize the probability amplitudes as the wave function on the lattice (*n*,*k*) to 1, as required by the definition of probability.

Thus, we can state the following proposition.

Proposition 1 (Mapping): *There exists a discrete mapping onto the wave function that solves the Schrödinger equation for a free particle via the Tartaglia triangle. Such a mapping* (18) *gives complex-valued probability amplitudes T(n,k) (eq.* (21)*) whose squares are the binomial coefficients (eq.* (20)*). These amplitudes can be normalized to 1 on a discrete (n,k) lattice.*

Proposition 1 gives the demonstration of the conjecture that was given in Section 7.2 of [1]. We note that the 2^n can be moved from the scalar product (26) to the wave function (25) without changing the conclusions.

What we have shown here is that the operator of time evolution in quantum mechanics does not represent a stochastic process per se but, in a discretized version of quantum mechanics on a lattice [9]-[10], one can recover the asymptotic of the binomial coefficients, representing the pdf of a walking drunk and, in the proper limit, a Brownian motion. Our conclusion is that time evolution (in discrete time domain) in quantum mechanics represents an underlying layer to a stochastic process being its square root. Also the complex valued Tartaglia amplitudes are the square root of the binomial coefficients except for the phase factor $e^{i\tilde{\phi}(n,k)}$. Thus quantum mechanics is a more fundamental theory than the stochastic process one. So, one can argue to do an inverse path and recover an upward layer to a given quantum system representing some kind of a different stochastic process. These ideas have been put in a fully mathematical framework in [11] where it was shown, after a proper redefinition of the Itō integral, that the extraction of the square root of a Wiener process gives a complex-valued stochastic process with a diffusional equation being the Schrödinger equation. Indeed, the following theorem can be proven true [11].

Square root theorem: *The square root of a stochastic process implies a diffusional process evolving with the Schrödinger equation.*

The square root in this case can be built by a generalization of the Itō integral and gives [11]

$$dX(t) = [dW(t) + \beta dt]^{\frac{1}{2}} = \left[\tfrac{1}{2} + \left|dW(t)\right| - \left(1 - \beta \cdot sign(dW(t)\right)dt\right]\Phi(t) \tag{27}$$

being *W(t)* a standard Wiener process, β a real drift parameter and $\Phi(t)$ a Bernoulli process with symmetric distribution having mean *(1+i)/2* and variance *i/2*. This process can be written down as

$$\Phi(t) = \frac{1-i}{2} sign(dW(t)) + \frac{1+i}{2}. \tag{28}$$

Now, by noting that *dW(t)=sign(dW(t))|dW(t)|* defining in this way this process, we can rewrite the square root of a Brownian motion into a more conventional way

$$dX(t) = [dW(t) + \beta dt]^{\frac{1}{2}} = \tfrac{1}{2}\Phi(t) + \tilde{\Phi}(t)dW(t) - \left(1 - \beta \cdot sign(dW(t)\right)\Phi(t)dt \tag{29}$$

being now $\tilde{\Phi}(t) = \frac{1+i}{2} sign(dW(t)) + \frac{1-i}{2}$. This Bernoulli process, representing the tossing of a coin, is the reason for the rotation of time from real to imaginary[5]. This can be understood by noting that this stochastic process produces the value i and 1 with probability 0.5 and so, it is the consequence of an extraction of a square root. But, in turn, this does not rotate time, rather it yields an imaginary diffusion coefficient that we can interpret as a rotated time variable. Indeed, a more general formula for any power of a Wiener process can be given (see [11]), and so, if e.g., we would have considered a Brownian motion at power $1/3$, we would have obtained an ordinary diffusion process instead.

3. NUMERICAL ANALYSIS

Our aim here is to give a numerical understanding of the particular solution of the Schrödinger equation (8), (14) and to see the graphical aspect of the discrete solution (eq. (25)) and the way the continuous solution maps on it (see eqs. (18)).

Firstly, let us take a look into the continuous domain solution of the Schrödinger equation and its geometrical meaning. We note that eq. (15) represents a four dimensional object being composed by a real and imaginary parts both depending on the position and time. Then, Figure 1 shows the real versus the imaginary parts of the solution (see eq. (15)) of the Schrödinger equation for a given value of *t*=0.499.

The depicted curve fits rather well with a logarithmic spiral

$$\rho = ce^{b\theta} . \tag{30}$$

Indeed, by considering the variable substitution of

$$x^2 \to k(t)\theta \tag{31}$$

where *k(t)* is a proper scaling factor, it can be shown the logarithmic spiral parameters to be

$$\begin{aligned} c &= \left(\frac{\frac{a^2}{\pi}}{a^4 + \frac{\eta^2 t^2}{m^2}} \right)^{\frac{1}{4}} \\ b &= -0.5k(t) \frac{a^2}{a^4 + \frac{\eta^2 t^2}{m^2}} \end{aligned} \tag{32}$$

These parameters are depicted in Figure 2. The continuous lines refer to the above equations, while the circle markers refer to the logarithmic spiral equation parameters derived by fitting the achieved results of the solution of the Schrödinger equation for several values of the time *t.*

[5] We were able to prove the existence of this stochastic process, the square root of a Brownian motion, using a simulation code running on Matlab® [17] giving also numerical support to this theorem.

Figure 3 shows the *Real[$\psi(x,t)$]* part versus the *Imag[$\psi(x,t)$]* part of the solution of the Schrödinger equation (15) for a given value of the space variable *x* (*x*=5 m.*)*.

Finally, in Figure 4 a three dimensional (3D) plot of the Schrödinger equation solution is given, varying the spatial coordinate at fixed time. The curve we obtained is strongly rensembling a Beltrami hypersphere[6], typical of an hyperbolic geometry. A Beltrami hypershphere has a tractrix[7] ([16]) as a section curve:

$$x = -\alpha \log\left[\frac{\alpha}{r} + \frac{\sqrt{\alpha^2 - r^2}}{r}\right] + \sqrt{\alpha^2 - r^2} \tag{33}$$

where *x* and *r* are the vaiables of the tractrix funxtion *x*=*f*(*r*) and α is a suitable parameter of the tractrix.

In order to prove this, we have taken a section of the surface of Figure 4 along *x* direction and we perfomed a fit (see Figure 5). The result is a tractrix equation with a polynomial correction

$$x = -\alpha \log\left[\frac{\alpha}{r} + \frac{\sqrt{\alpha^2 - r^2}}{r}\right] + \sqrt{\alpha^2 - r^2} + c \cdot r + d \cdot r^2 + e \tag{34}$$

The tractrix appears as time evolves and fits worsen at the initial times while improves with increasing time. This shows that the initial hypothesis of Beltrami hypersphere is rather well satisfied at increasing time.

By considering the discrete case of eq. (22), the following figures show how one recovers the behavior similar to the continuous case of as it should. Due to the effect of mapping (18), the structure of the curves is maintained proving the consistency of the mathematical structure we have devised. Due to the fast increasing of the binomial coefficients and the nonlinear nature of the mapping, in the discrete case, the Beltrami sphere is somewhat flattened but the geometric structure is still there, sampled at some points in this discrete space and time lattice *(n,k)*. Figure 6 shows the *x(n,k)* and *y(n,k)* of eq. (22) where *k* runs and *n* is set to *n*=500.

Figure 7 is the analogous in the discrete lattice *(n,k)* of Figure 4.

We also give an analysis of the modulus $|T(n,k)|$ in Figure 8 (the color code accounts for the amplitude) and phase $\tilde{\phi}(n,k)$ in Figure 9 (the color code accounts for the unwrapped phase value) of the Schrödinger equation solution in the discrete space time lattice *(n,k)*. These figures give a feeling of how the Schrödinger solution changes passing from the continuous (*x*,*t*) case to the discrete (*n*,*k*) lattice, being the map non-linear (see eqs. (18)).

Figure 9 depicts the phase $\tilde{\phi}(n,k)$ of eq. (24) versus n and k, taking the absolute value of the phase amplitude and coded in colour.

But, as seen in Figure 10, the spreading of the wave packet $|x(n,k)|^2 + |y(n,k)|^2$ from eq. (22) appears identical to the continuous case. Infact we have the binomial coefficients of eq. (20).

[6] The Beltrami hypersphere is a two-dimensional surface of constant negative curvature as opposite to a classical sphere with positive Gauss curvature. Just as the sphere has at every point a positively curved geometry of a dome, the whole pseudo-sphere has at every point the negatively curved geometry of a saddle (From: Wikipedia).

[7] In [16] it is shown how the tractrix equation is the solution of the classical pursuit problem known also as the "dogleg curve" due to the path made by a dog following its master.

4. CONCLUSIONS

The conjecture that was made in [1] concerning a potential connection between the solution of the classical Schrödinger equation and the Tartaglia triangle has been proved.

We were able to show how a map exists between quantum mechanics and Tartaglia-Pascal triangle. The striking aspect of this result is that it appears rather counterintuitive as the well-known Tartaglia-Pascal triangle appears in the Schrödinger solution through its square root. This motivated one of us (M.F.) to propose a reformulation of the Itō calculus and to prove a quite general result: Quantum mechanics is the square root of a stochastic process [11].

Numerical analysis produced an interesting relationship between the geometry of the Beltrami sphere and the evolution in time of the Schrödinger wave function and showed how the non-linear map we devised preserves essentially unchanged the behaviour of the wave-packet during its evolution, given the connection with the binomial coefficients and so, the Tartaglia-Pascal triangle.

These ideas showed a fruitful connection between quantum mechanics and stochastic processes that was somewhat unexpected. In retrospective, it appears clear how difficult could have been to display this kind of understanding.

On the other side, this new view on stochastic processes could pave the way to a new look of the Dirac equation, a relativistic generalization of the Schrödinger equation. New applications for signal processing can be found. Some new ideas in the analysis complex networks with Brownian motion propagating across the network could be envisaged. Since random walk problems might be also studied on complex network (ubiquitous in today world), we express the hope that the results published in this paper will contribute to solve such interesting problems. In any case, reformulation through new mathematical tools are generally rather fruitful and we hope this is the case also here.

5. APPENDIX A

The Schrödinger and the heat equations have a common mathematical structure that makes them easy to manage using a Fourier transform. Indeed, let us consider the general parabolic differential equation

$$\frac{\partial\phi}{\partial t} = k\frac{\partial^2\phi}{\partial t^2} \tag{35}$$

with $k \in \Re$ for the heat equation and $k = \frac{i\eta}{2m}$ for the Schrödinger equation. This equation is linear and can be treated with a Fourier transform.

So, let us define the mode *p*

$$\hat{\phi}_p = \Im(\phi) = \int_{-\infty}^{+\infty}\phi(x)e^{-ipx}dx \tag{36}$$

for the Fourier transform in the space coordinate. The given equation becomes

$$\frac{\partial\hat{\phi}_p(t)}{\partial t} = -kp^2\hat{\phi}_p(t) \tag{37}$$

that can be immediately solved to give

$$\hat{\phi}_p(t) = \hat{\phi}_p(0)e^{-kp^2t} \tag{38}$$

Now, depending on *k* we get two completely different behaviours: for a real *k* this solution is decaying in time and so, represents an irreversible behavior of the solution for the mode *p*. But the behavior of the Schrödinger equation is quite different due to the presence of the $i=\sqrt{-1}$ factor. In this case, also the solution with a reversed time is physically meaningful and can be also obtained but taking the conjugate of the given solution. Indeed, this is exactly the effect of the time reversal operator in quantum mechanics and the Schrödinger equation is invariant for time reversal, contrarily to the case of the heat equation.

Now we are in a position to get the kernel solution for both equations. If we have a point-like source centered at $x = 0$, then its Fourier transform is a constant and we have to evaluate the integral

$$\phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ipx}e^{-kp^2t}dp \tag{39}$$

This is a Gaussian integral that can be readily evaluated to give the kernel for both equations. In the case of the Schrödinger equation we have to cope with Fresnel integrals but the result is the same and there is no convergence problem ([18]). Indeed, in the current literature (e.g. [18]) these integrals are always dubbed "Gaussian" and are generally straightforward to evaluate.

Finally, let us consider the case when the source is not point-like. So, we will get

$$\phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi_p(0)e^{-ipx}e^{-kp^2t}dp \tag{40}$$

that can be seen as the Fourier transform of a convolution between the kernel and the initial value of the solution. In this way we have recovered all the equations given in (5)-(6) and (10)-(12) of this paper.

We are now in a position to show the way a Gaussian wavepacket spreads during its time evolution. For our aims it is enough to take $\phi_p(0)$ as a Gaussian and we get

$$\phi x,t) = \left(\frac{a^2}{2\pi^3}\right)^{\frac{1}{4}}\int_{-\infty}^{+\infty} e^{\left\{-(ka)^2-\frac{i\eta k^2 t}{2m}+ikt\right\}}dk = (2\pi)^{-\frac{1}{4}}\left(a^2+\frac{i\eta t}{2ma}\right)^{-\frac{1}{2}} e^{-\frac{x^2}{4a^2+\frac{2i\eta t}{m}}} \tag{41}$$

The position probability density of the Gaussian wavepacket is then

$$|\phi(x,t)|^2 = \left\{2\pi\left[a^2+\frac{\eta^2t^2}{4m^2a^2}\right]\right\}^{-\frac{1}{2}} e^{-\frac{x^2}{2\left[a^2+\frac{\eta^2t^2}{4m^2a^2}\right]}} \tag{42}$$

that is eq.(15) using the given definition for the Heisenberg time.

6. ACKNOWLEDGEMENTS

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FIGURES

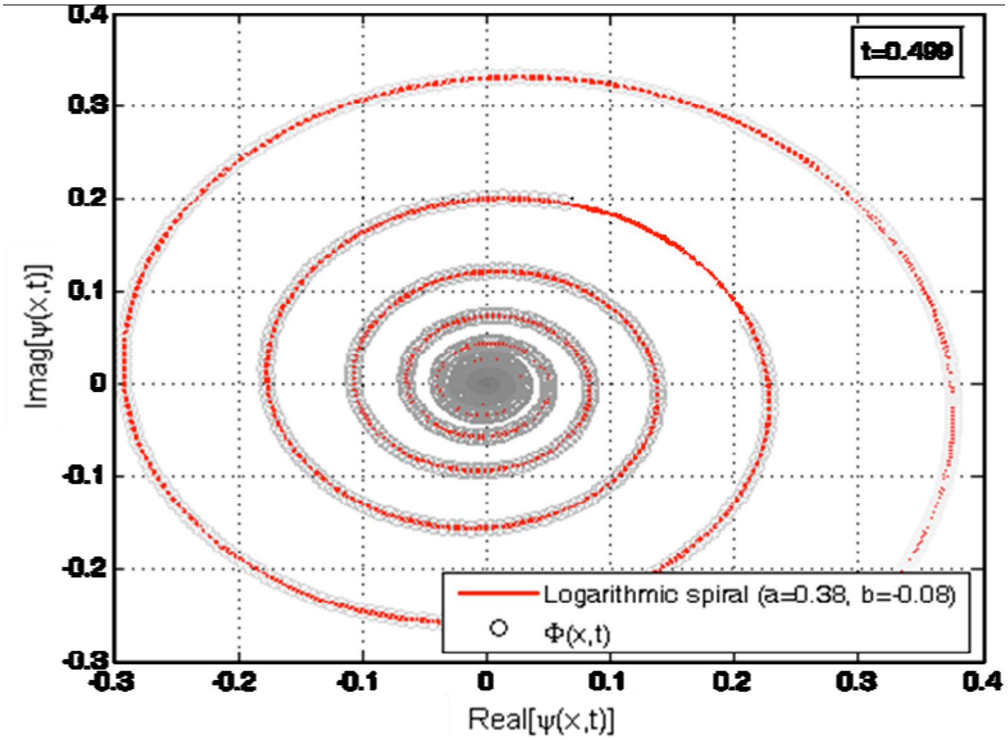

Figure 1 - *Real[ψ(x,t)]* versus *Imag[ψ(x,t)]* of the solution of the Schrödinger equation for t=0.499 m.

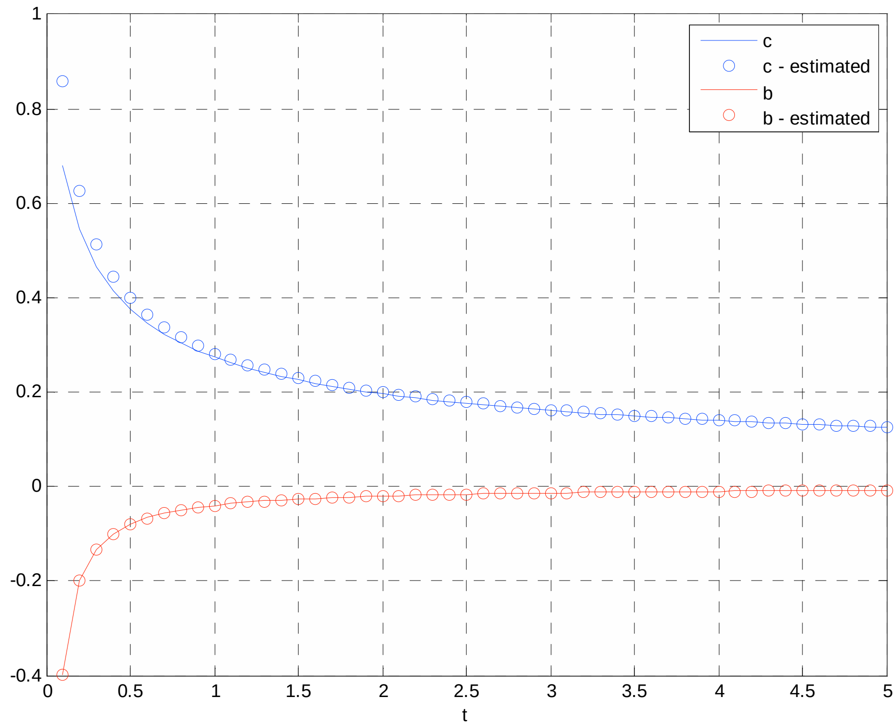

Figure 2 - Polar spiral parameters.

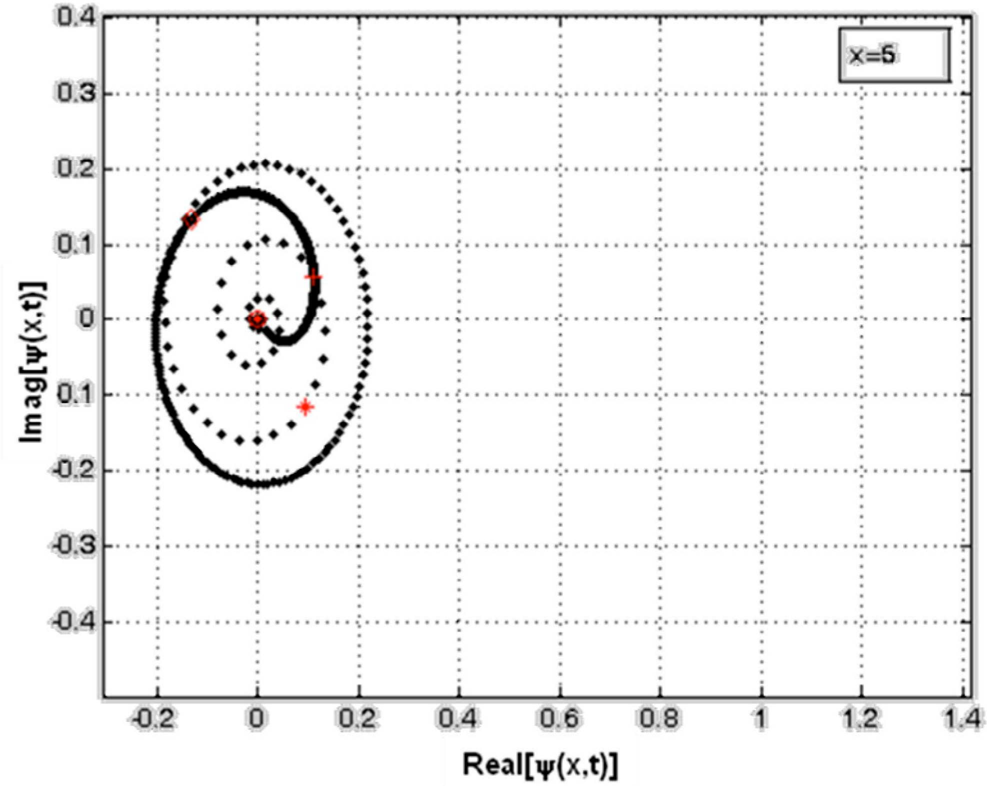

Figure 3 - *Real[ψ(x,t)]*versus *Imag[ψ(x,t)]* of the solution of the Schrödinger equation for *x*=5m.

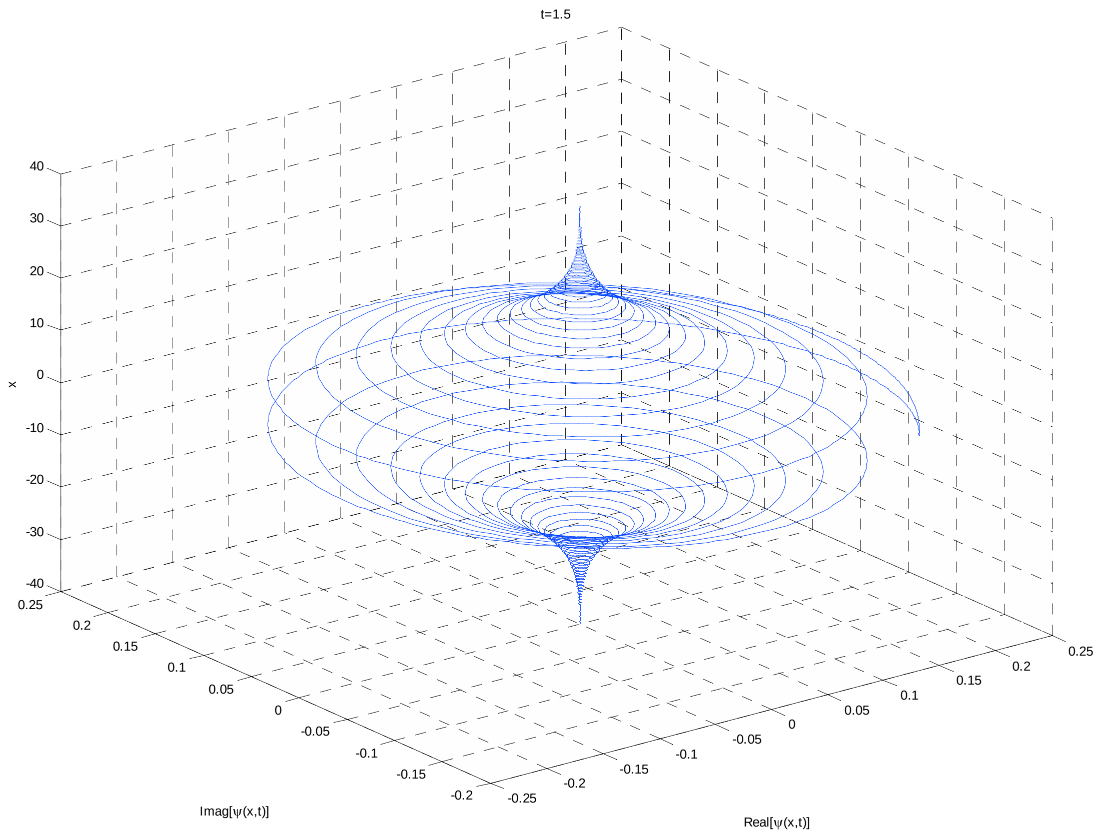

Figure 4 - 3D plot of *Real[ψ(x,t)]* (x axis), *Imag[ψ(x,t)]* (y axis), and *x* (z axis) given *t*.

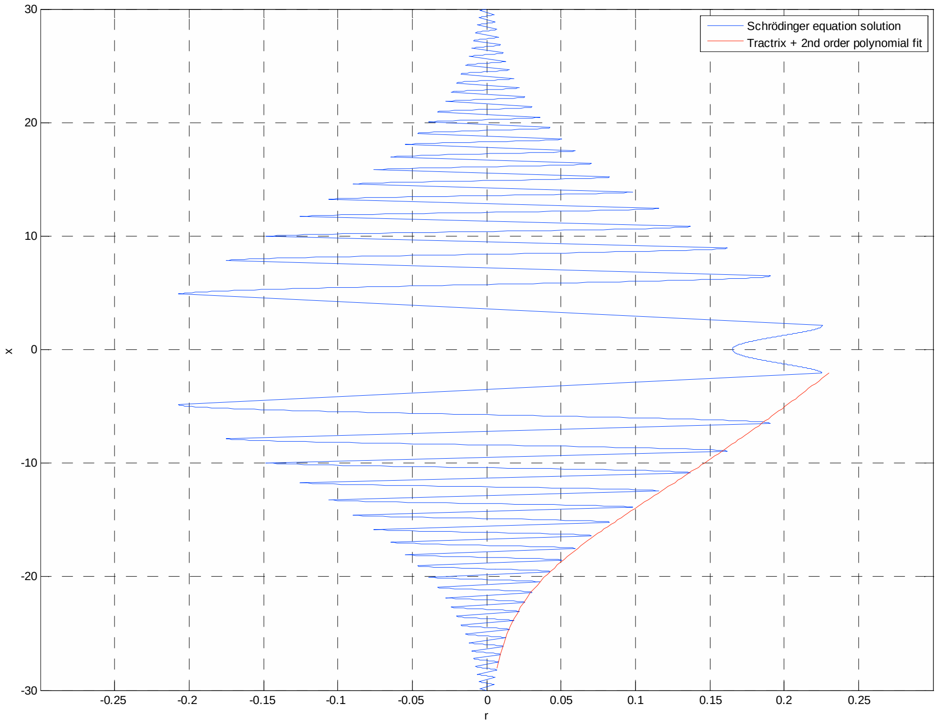

Figure 5 - Schrödinger solution (blue line) envelope fitted with a tractrix (red line) with a 2nd order polynomial correction.

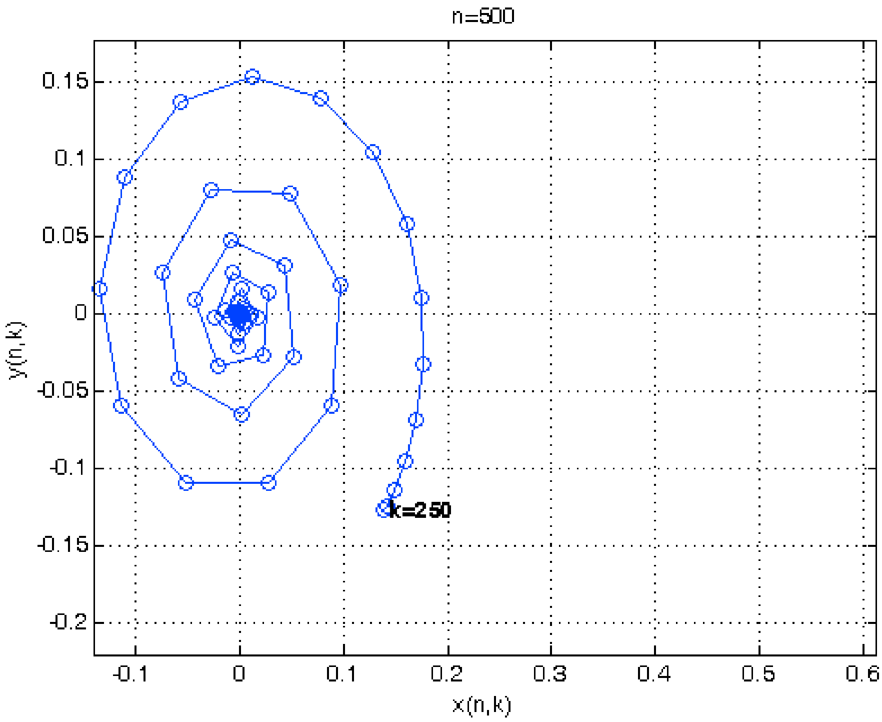

Figure 6 - Logarithmic spiral on the discrete space-time lattice.

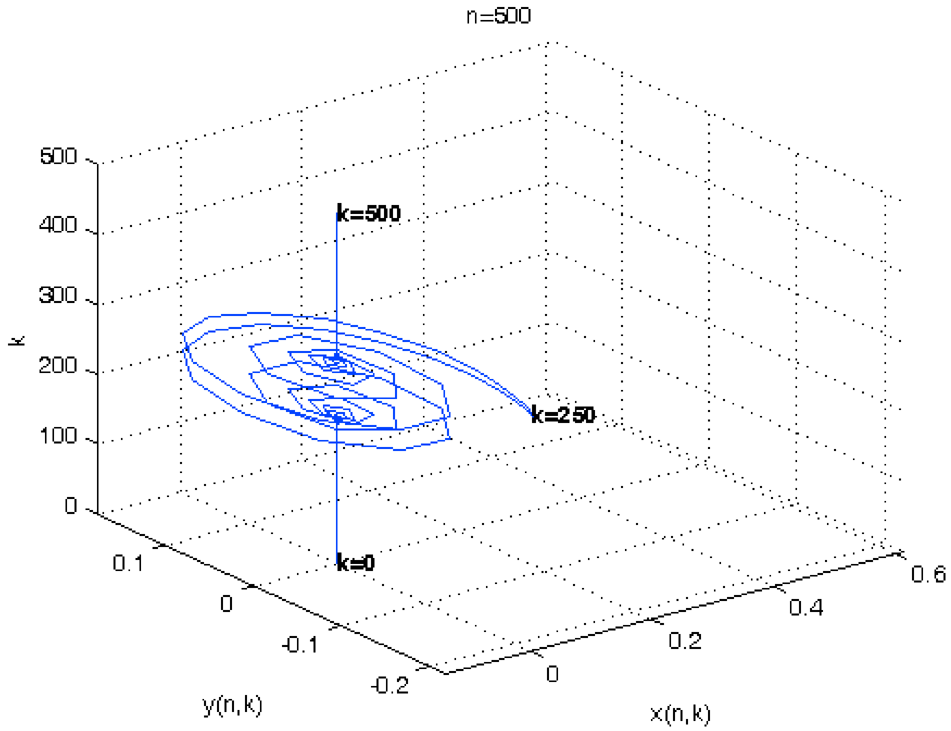

Figure 7 – Evolution of *x(n,k)* and *y(n,k)* of eq. (22) as *k* is running and *n* is equal to 500.

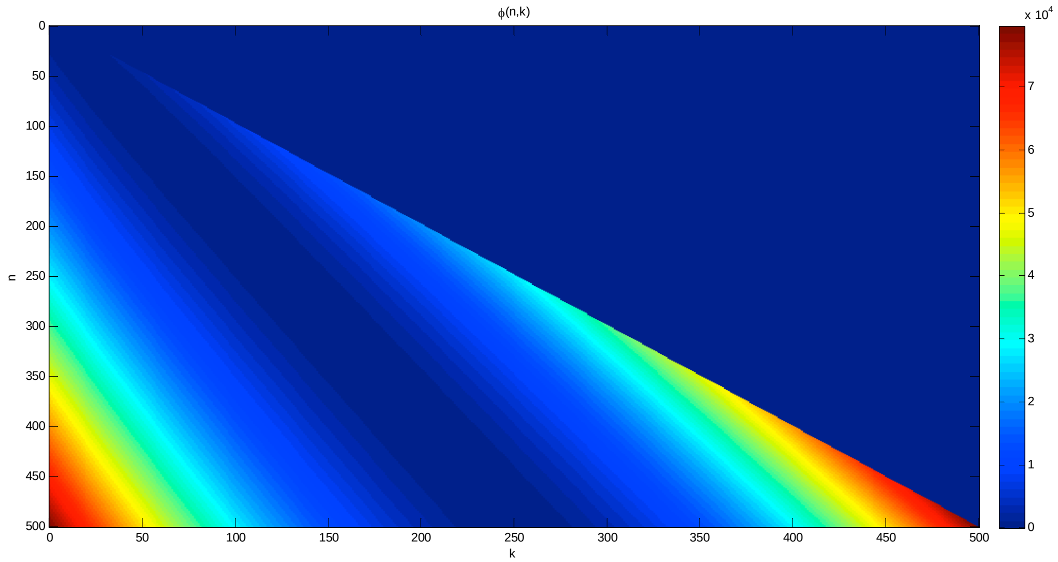

Figure 8 – Amplitude value of the Schrödinger equation solution of the discrete (n,k) lattice (color code represents the amplitude).

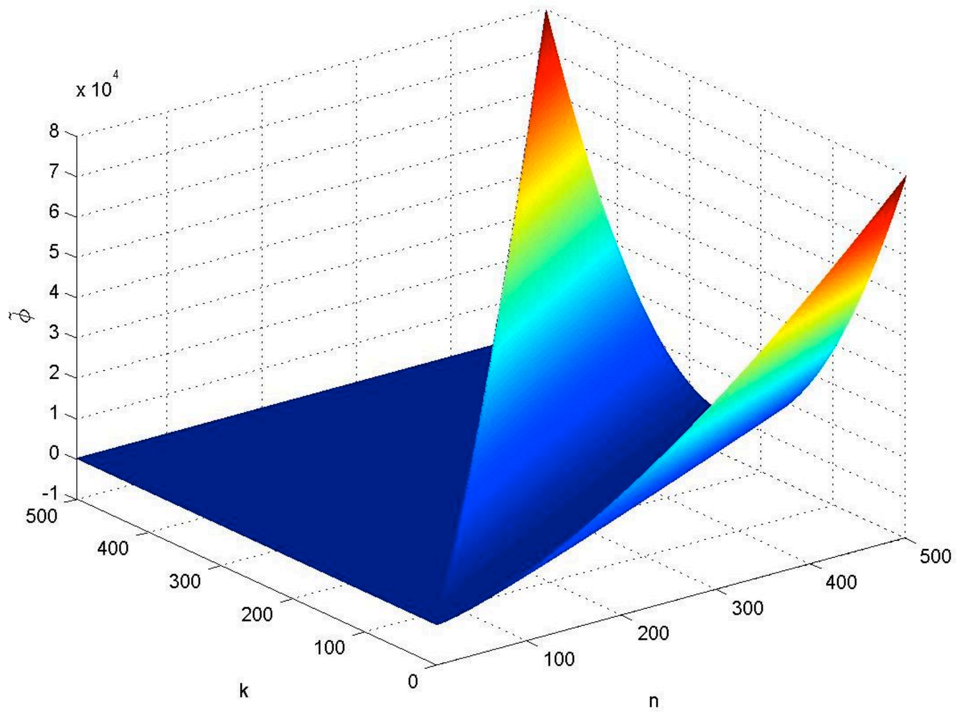

Figure 9 – Phase value of the Schrödinger equation solution of the discrete (n,k) lattice (color code represents the unwrapped phase).

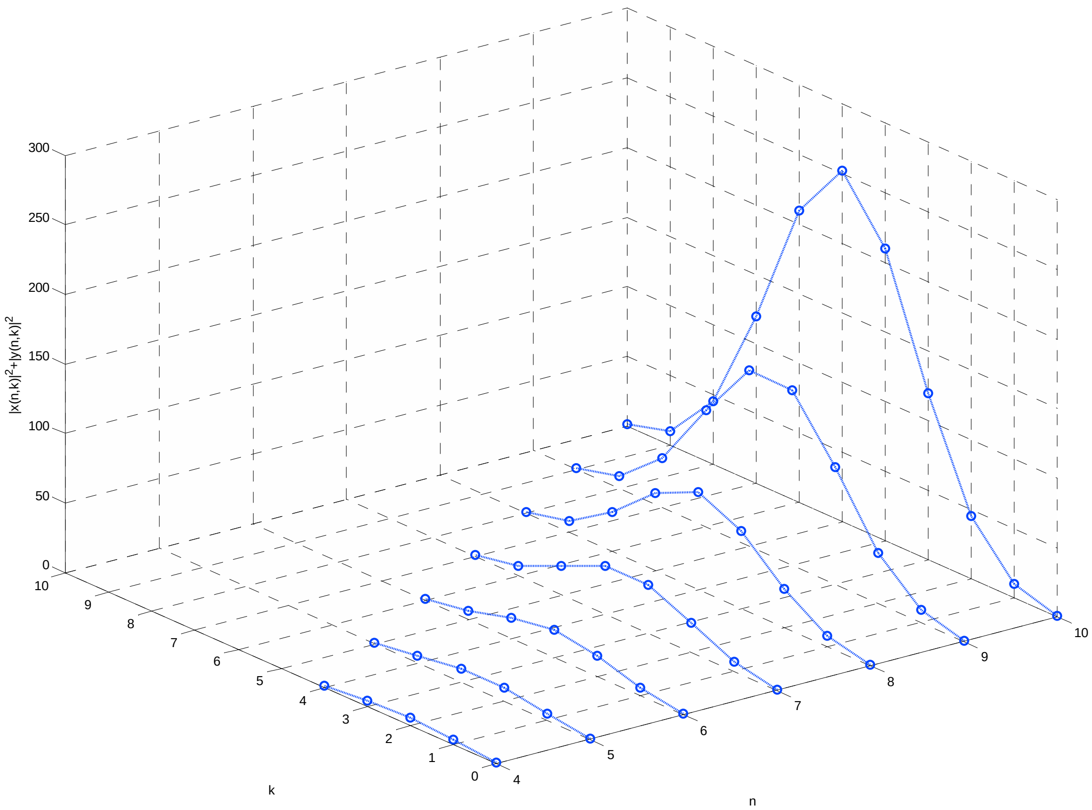

Figure 10 - Spreading of the wave packet in the discrete space-time lattice.